ON REALIZING SYMMETRIC 3-POLYTOPES*

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ABSTRACT

We prove the analogue of Eberhard's Theorem for symmetric convex 3-polytopes with a 4-valent graph, and disprove a conjecture of the late T. Motzkin about realizing symmetric convex 3-polytopes so that all of their geodesics are in planes.

The purpose of this paper is to prove the following:

THEOREM 1. *Every sequence* $(p_k \mid 4 \neq k \geqq 3)$ *of non negative* even *integers satisfying* $\sum_{k \geqq 3}(4-k)p_k = 8$ *is 4-realizable by a centrally symmetric, line symmetric and plane symmetric 3-polytope.*

THEOREM 2. *For every* $n \geqq 4$ *there exists a centrally symmetric 3-polytope* P_n, *having a 4-valent graph and n simple closed self-antipodal geodesics* $\alpha_1, \cdots, \alpha_n$, *such that if a polytope* P'_n *is combinatorially equivalent to* P_n *with corresponding geodesics* $\alpha'_1, \cdots, \alpha'_n$, *then no one of the geodesics* $\alpha'_2, \cdots, \alpha'_n$ *of* P'_n *is in a plane.*

1. Theorem 1

A sequence $(p_k \mid 4 \neq k \geqq 3)$ of non negative integers is said to be *4-realizable* if there exists a value for p_4 and a 3-polytope P (= the convex hull of a finite set of points in E^3 with non empty interior) such that P has a 4-valent graph and p_k k-gons in its boundary cell-complex, for all $k \geqq 3$. For additional definitions, see [3].

Theorem 1 is a relative to Eberhard's Theorem [2], and it was suggested by B. Grünbaum ([3], p. 269, # 8). For related results, see, in addition, [4], [5] and [9].

E. Jucovič mentioned, in a private communication, that he had independently proved Theorem 1 (with one exceptional case).

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Note that the conditions "all p_k even and $\Sigma_{k\geqq 3}(4-k)p_k = 8$" are necessary for the 4-realizability of $(p_k | 4 \neq k \geqq 3)$ by a symmetric 3-polytope.

We need the following:

LEMMA 1. *If F is a face of a 3-polytope P, F is contained in the plane π and $T: E^3 \to \pi$ is the orthogonal projection, then there exists a polytope P', combinatorially equivalent to P, such that F is a face of P' and $(T|_{P'})^{-1}[\mathrm{Bd}\, F] = \mathrm{Bd}\, F$.*

PROOF. Suppose, without loss of generality, that $(0,0,0) \in \mathrm{Int}\, F$,

$$\pi = \{(x,y,z) | x = 0\} \text{ and } P \subset \{(x,y,z) | x \leqq 0\}.$$

A supporting hyperplane of P, determined by a face ($\neq F$) of P, meets the positive x-ray in at most one point; so let $\alpha > 0$ be small enough such that $(\alpha,0,0)$ is not strictly separated from P by any of these hyperplanes.

The transformation S, defined by

$$S(x,y,z) = \left(\frac{\alpha x}{\alpha - x}, \frac{\alpha y}{\alpha - x}, \frac{\alpha z}{\alpha - x}\right)$$

is a projective transformation of E^3 (in fact, S is a $1-1$ projective transformation when properly applied to P^3, the real projective 3-space, with $E^3 \subset P^3$). $S|_\pi$ is the identity (pointwise!) and $S[\{(x,y,z) | x = \alpha\}]$ is the plane at infinity.

Therefore $S(P) = P'$ is a 3-polytope, combinatorially equivalent to P, and $F = S(F)$ is a face of P'.

Since no supporting hyperplane of P (except for π), determined by a face of P, meets the *closed* segment $(0,0,0) - (\alpha,0,0)$, it follows that their image under S do not meet the ray $\{(\alpha,0,0) | \alpha \geqq 0\}$ plus the point at infinity on this ray — $S(\alpha,0,0)$. Therefore they must all meet $\{(\beta,0,0) | \beta < 0\}$.

It follows immediately that $T(P') = F$ and that $(T|_{P'})^{-1}[\mathrm{Bd}\, F] = \mathrm{Bd}\, F$.

REMARK. Lemma 1 can be extended to m-polytopes in the obvious way; the corresponding transformation S becoming

$$S(x_1,\cdots,x_m) = \left(\frac{\alpha x_1}{\alpha - x_1}, \frac{\alpha x_2}{\alpha - x_1}, \frac{\alpha x_m}{\alpha - x_1}\right).$$

Compare Lemma 1 with [3], p. 82–83 (adjoining polytopes) and [7], p. 191 (the transformation τ_p).

PROOF OF THEOREM 1. Let $(p_k | 4 \neq k \geqq 3)$ be given such that all p_k are even and $\Sigma_{k\geqq 3}(4-k)p_k = 8$.

Concerning the sequence $(q_k \mid 4 \neq k \geqq 3)$, defined by $q_k = \frac{1}{2} p_k$ for all k we have the following

CLAIM. *There exists a 3-polytope Q, having a face F_1 which is an m-gon for some $m \geqq 5$, such that*

i) *Q has q_k k-gons, for all $k \geqq 3$ and $k \neq 4, m$, and it has $q_m + 1$ m-gons,*

ii) *all vertices of Q are 4-valent, except for those of F_1 which are 3-valent,*

iii) *m is even and F_1 is centrally symmetric,*

iv) *F_1 and Q satisfy the same condition that F and P' satisfy in Lemma 1.*

PROOF OF THE CLAIM. We first construct a 3-connected planar graph G (see Fig. 1) that has q_k k-gons for all $k \geqq 3$ and $k \neq 4, m$, and has $q_m + 1$ m-gons, for some m (compare figure 13.3.3 in [3], figure 2 in [4] and figure 3 in [9]):

For each k, $k \geqq 5$, there are q_k k-gons, arranged along the diagonal, such that near each such a k-gons there are $k - 4$ triangles.

Since $q_3 = 4 + \Sigma_{k \geqq 5}(k-4)q_k$, the four additional triangles are located outside the main square, one near each vertex of the square.

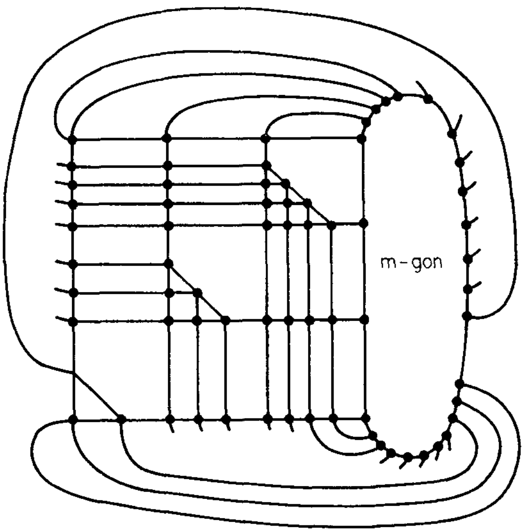

Fig. 1

Since G has exactly m vertices of odd order, it elementarily follows that m is even.

Using Barnette-Grünbaum's version [1] of Steinitz's Theorem [8], where the m-gon is preassigned as a regular m-gon F_1, we get a polytope Q_1 satisfying the corresponding properties (i) (ii) and (iii). The promised polytope Q having F_1 as a face is obtained by applying Lemma 1 to Q_1 and F_1.

This completes the proof of the claim.

To complete the proof of Theorem 1, let Q^* be

1) the centrally symmetric image of Q through the center of F_1, if P is to be centrally symmetric;

or 2) the line symmetric image of Q through a line of symmetry of F_1, if P is to be line symmetric,

or 3) the plane symmetric image of Q through the plane that contains F_1, if P is to be plane symmetric.

$P = Q \cup Q^*$ is the required polytope: its convexity follows from property (iv) of Q; since the face F_1 of Q disappears in P, P has a 4-valent graph and it has $p_k = 2q_k$ k-gons for all $4 \neq k \geqq 3$.

Theorem 1 has been established.

REMARK. If "line symmetric" is deleted from Theorem 1, a simpler proof can be given, using Theorems 5 and 6 ([3], p. 245–6) as follows:

Let G' be the graph, isomorphic to G of Fig. 1, such that the corresponding m-gon F_1 is the unit disc in E^2, with the convex hull of the vertices of F_1 being a regular m-gon. Let f, g: $E^2 - 0 \to E^2$ be defined by

$$f(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right) \text{ and } g(x,y) = (-x,-y).$$

Since $\mathrm{Bd}\, F_1 = G' \cap gf(G') = G' \cap f(G')$, it follows that both $G^* = G' \cap gf(G')$ and $G^{**} = G' \cap f(G')$ are 4-valent 3-connected planar graphs having each p_k k-gons for all $4 \neq k \geqq 3$. The mapping gf is an involution on G^* such that for each vertex v of G^*, v and $gf(v)$ are separated by a circuit in G^*; it follows from Grünbaum's Theorem 5 ([3], p. 245) that G^* is the graph of a centrally symmetric polytope. The mapping f is an involution on G^{**} such that for each face A of G^{**}, A and $f(A)$ have opposite orientations; it follows from Grünbaum's Theorem 6 ([3], p. 246) that G^{**} is the graph of a plane symmetric polytope, as promised.

2. Theorem 2

A *geodesic path* on a 3-polytope P, having a 4-valent graph $G(P)$ is a collection of edges $E_1, \cdots, E_k$ of $G(P)$ such that E_i and E_{i+1} have a common vertex but do not lie on a common face of P, for all i, $1 \leqq i \leqq k-1$ (see [3], p. 239); a geodesic path is *closed* if, in addition, E_1 and E_k have a common vertex but not a common face; it is *simple* if it does not intersect itself.

T. Motzkin asked the following [6]: Suppose P is a centrally symmetric 3-polytope with a 4-valent graph, having simple closed self-antipodal geodesics. Does there exist a polytope P', combinatorially equivalent to P, having each one of its geodesics in a plane?

A curve α on a centrally symmetric 3-polytope P with center 0 is said to be *self-antipodal* if α is centrally symmetric with center 0.

Our Theorem 2 is clearly a sharp negative answer to Motzkin's question.

For the proof of Theorem 2 we need the following

LEMMA 2. *For every $n \geqq 4$ there exists a 3-connected graph G_n in the plane E^2 having the following properties*

i) *G_n has a regular $(2n-2)$-gon F_n such that the vertices of G_n are 4-valent, except of the vertices of F_n which are 3-valent.*

ii) *The edges of G_n decompose into n simple arcs $\beta_1, \cdots, \beta_n$, where $\beta_1 = \mathrm{Bd}F_n$ and β_i is a simple geodesic connecting antipodal vertices of F_n, for all i, $2 \leqq i \leqq n$.*

iii) *G_n has a k-gon H_n such that for all i, $2 \leqq i \leqq n$, $\beta_i \cap H_n$ contains at least two edges.*

iv) *H_n meets F_n in an edge of G_n.*

PROOF. The proof is by induction on n. Starting with $n = 4$, G_4 is described in Fig. 2.

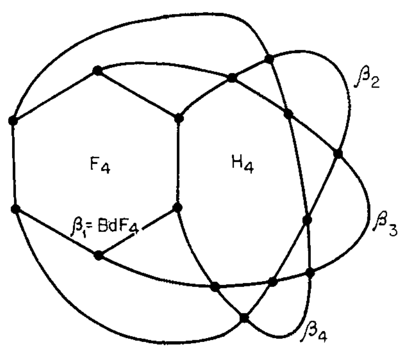

Fig. 2

Suppose the assertion is true for n, $n \geqq 4$. To show that it is true for $n+1$, let G_n, F_n, H_n, $\beta_1, \cdots, \beta_n$ satisfy the conditions of Lemma 2.

By (iv) $H_n \cap F_n$ is an edge E_n of G_n. Let x be an interior point of an edge of Bd F_n, adjacent to E_n (see Fig. 3), and let y be the antipodal of x, with respect to F_n.

Let β_{n+1} be a path from x to y lying outside F_n (except for its endpoints), and such that at each vertex of H_n, except for the vertices of E_n, β_{n+1} alternatingly cuts in and out of H_n, while crossing edges of G_n (see solid curve in Figure 3); β_{n+1} is taken near $\mathrm{Bd}\, H_n \cup \mathrm{Bd}\, F_n$.

β_{n+1} meets and crosses alternatively either all the four edges that meet at a vertex of H_n or else only the two edges of H_n among the four which meet at a vertex of H_n, in the fashion described in Fig. 3.

The graph G_{n+1} is isomorphic to $G_n \cup \beta_{n+1}$ under a homeomorphism that takes F_n onto a regular $2n$-gon F_{n+1} (where $G_n \cup \beta_{n+1}$ means the following: add all points of $\beta_{n+1} \cap G_n$ as vertices, subdividing the edges on which each of them lies, and add all arcs of $\beta_{n+1} - G_n$ as new edges). In G_{n+1}, H_{n+1} is taken to to be the closure of that connected component of $H_n - \beta_{n+1}$ that contains E_n.

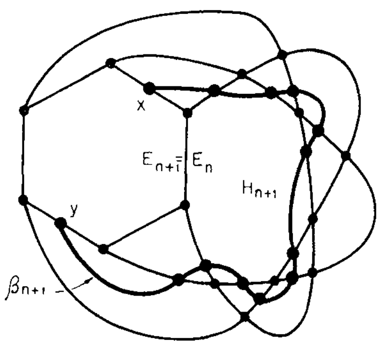

Fig. 3

It is obvious that G_{n+1}, F_{n+1}, $\beta_1, \cdots, \beta_{n+1}$ satisfy conditions (i) and (ii). Since $n \geqq 4$, β_2, β_3 and β_4 meets H_n as described in (iii), hence H_n has at least six edges (except E_n), and therefore β_{n+1} enters H_n at least twice, hence $\beta_{n+1} \cap H_{n+1}$ contains at least two (disjoint) edges. Since β_{n+1} meets all the edges of H_n (except E_n and possibly another edge of H_n, incident to E_n), it follows that every edge of H_n (except possibly the two, as before) is divided into two, one of which becoming an edge of the new H_{n+1}. Therefore condition (iii) is satisfied. Since E_n is untouched, $E_{n+1} = H_{n+1} \cap F_{n+1} = H_n \cap F_n = E_n$, hence condition (iv) is satisfied. Clearly, G_{n+1} is 3-connected if G_n is.

This completes the proof of Lemma 2.

PROOF OF THEOREM 2. For every $n \geqq 4$, let G_n, F_n, H_n, $\beta_1, \cdots, \beta_n$ be as given in Lemma 2.

As in the proof of Theorem 1, let Q_n be a 3-polytope in E^3 having G_n for its graph and such that the face (corresponding to) F_n is a regular $(2n-2)$-gon centered at the origin; Q_n can be so chosen, using Lemma 1, that the inverse of the projection of Q_n into the plane containing F_n is $1-1$ on $\mathrm{Bd}(F_n)$.

Let $P_n = Q_n \cup (-Q_n)$. P_n is a centrally symmetric 3-polytope with a 4-valent graph. The geodesics $\alpha_1, \cdots, \alpha_n$ of P_n are given by: $\alpha_1 = \beta_1 = \mathrm{Bd}\, F_n$ and $\alpha_i = \beta_i \cup (-\beta_i)$ for all i, $2 \leqq i \leqq n$. (To avoid confusion, we use the same name for objects in G_n and in $\mathrm{Bd}\, Q_n$).

Clearly each geodesic of P_n is simple closed and self-antipodal.

Suppose that there exists a polytope P_n', combinatorially equivalent to P_n, such that a geodesic α_{i_0}' corresponding to α_{i_0} of P_n, for some i_0, $2 \leqq i_0 \leqq n$, is in a plane R. By property (iii) of G_n, $\alpha_{i_0} = \beta_{i_0} \cup (-\beta_{i_0})$ meets the 2-face H_n' of P_n', corresponding to H_n of P_n, in at least two edges. Therefore $H_n' \subset R$.

Since α_{i_0} is a geodesic, $\alpha_{i_0} \neq \mathrm{Bd}\, H_n'$, hence there exists a vertex $V \in \alpha_{i_0}$ with $V \notin \mathrm{Bd}\, H_n'$. $V \in \alpha_{i_0} \subset R$, hence $V \in P_n' \cap R$, and $H_n' \subsetneqq P_n' \cap R$.

This is a contradiction to the well-known property that a hyperplane R containing a 2-face H_n' of a 3-polytope P_n' is a supporting hyperplane and $H_n' = P_n' \cap R$ (see [3]). Hence no such a P_n' exists and Theorem 2 has been proved.

3. 3-valent Polytopes.

A sequence $(p_k \mid 6 \neq k \geqq 3)$ of non negative integers is said to be 3-realizable if there exists a value for p_6 and a 3-polytope P such that P has a 3-valent graph and p_k k-gons, for all $k \geqq 3$.

It is well known that a necessary condition for the 3-realizability of

$$(p_k \mid 6 \neq k \geqq 3)$$

by a centrally symmetric 3-polytope is that all of the p_k's be even and $\sum_{k \geqq 3}(6-k)p_k = 12$, see ([3], p. 253).

The following is B. Grünbaum's ([3], p. 269, # 8)

CONJECTURE 1. Every sequence $(p_k \mid 6 \neq k \geqq 3)$ of non negative *even* integers satisfying $\sum_{k \geqq 3}(6-k)p_k = 12$ is 3-realizable by a centrally symmetric 3-polytope.

Let $Q(m)$, $m \geqq 3$, be the following statement:

"Every sequence $(p_k \mid 6 \neq k \geqq 3,\ p_5 \geqq m$ and $p_m \geqq 1$ if $m \neq 5$, otherwise $p_5 \geqq 6)$ of non negative integers satisfying $\sum_{k \geqq 3}(6-k)p_k = 12$ is 3-realizable by a 3-polytope Q, such that an m-gon of Q is surrounded by pentagons only."

It can be easily proved, using ideas similar to the previous ones, that Conjecture 1 holds if $Q(m)$ is true for some even m, $m \geqq 4$. However, as was remarked by the referee and by Professor B. Grünbaum, $Q(m)$ is false for all $m \not\equiv 0 \pmod 6$;

QUESTION. Is $Q(m)$ true for some $m \equiv 0 \pmod 6$?

The author would like to thank Professor Branko Grünbaum and the referee for their many remarks and criticisms.

Added in proof: Our Theorem 2 is related to the existence of non-strechable simple arrangements of pseudolines in the plane, see B. Grünbaum's "Arrangements of Hyperplanes", Proc. Second Louisiana Conference on Combinatorics and Graph Theory, Baton Rouge, March, 1971.

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